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SAMPLED-DATA SYSTEM DESIGN WITH A POTENTIAL-PLANE ANALOG OF THE LOG Z-PLANE

by

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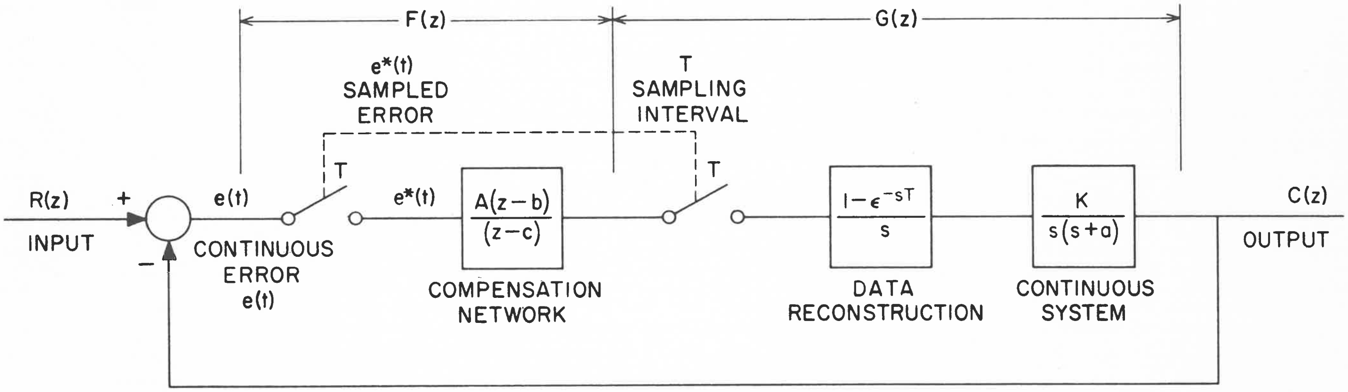

A SAMPLED-DATA CONTROL SYSTEM

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SAMPLED-DATA SYSTEM DESIGN WITH A POTENTIAL-PLANE ANALOG OF THE LOG Z-PLANE

INTRODUCTION

Extensive use of digital computers in the past decade has established a definite trend toward control systems that operate with variables which are in the form of a sequence of numbers. These variables are usually quantized in amplitude and available only at specific, equally spaced instants of time. In comparison, the continuous system has variables that are continuous functions of time. In the systems using discrete variables, the data appear as sequences of numbers or pulses and these data are valid only at the sampling instants. This type of system is usually called a sampled-data system.

Several methods of analyzing and synthesizing sampled-data systems have been discussed in the literature. One popular and effective method uses the Z-transform and root-locus plots of the closed loop system on a geometric analog of the z-plane. This paper shows how sampled-data problems can be solved with a potential-plane analog of the log z-plane. This method combines the advantages of fast solutions from a potential-plane analog and easy visualization of system changes on the log z-plane.

THEORY

One of the fundamental decisions to be made concerning a sampled-data system is what sampling period or frequency is to be used. Shannon has found that the theoretical minimum sampling frequency that can be used without loss of information is equal to twice the value of the highest frequency of interest in the data to be sampled. However, the minimum sampling frequency is based upon the use of an unattainable ideal, low-pass filter to reconstruct the sampled data, so practical systems use a sampling frequency which is usually much higher than the theoretical minimum.

Figure 1 shows a block diagram of a typical sampled-data feedback control system. The effect of the sampler is to convert a continuous function of time $\epsilon(t)$ into a sequence of pulses $\epsilon^*(t)$ whose amplitudes are the same as $\epsilon(t)$ at the sampling instants.

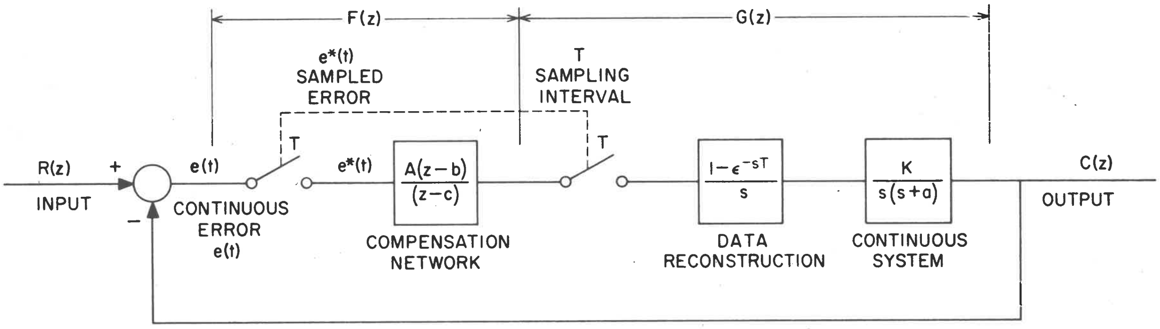


FIGURE I A SAMPLED-DATA CONTROL SYSTEM

The relationship between any continuous function $f(t)$ and the corresponding sampled function $f^*(t)$ has been derived by several authors (1,3,4) and only the results will be repeated here. If the assumption is made that the width of the sampling pulse is small compared to the sampling interval T, the mathematics can be simplified by the use of the impulse function $\delta(t)$.

For this case,

$$f^*(t) = f(t)\left[\sum_{n=0}^{\infty} \delta(t-nT)\right] \tag{1}$$

Since the impulse is very narrow, the actual value of $f(t)$ is $f(nT)$, the value of the function at the sampling instant. Thus Equation (1) can be written as

$$f^*(t) = \sum_{n=0}^{\infty} f(nT)\,\delta(t-nT) \tag{2}$$

If the Laplace transform is taken, term by term, the result is

$$F^*(s) = \sum_{n=0}^{\infty} f(nT)\,\epsilon^{-nTs} \tag{3}$$

It has been shown that an equivalent expression for $F^*(s)$ can be obtained with the aid of a Fourier series for $\delta(t-nT)$ which is

$$F^*(s) = \frac{1}{T}\sum_{n=-\infty}^{+\infty} F(s+nj\omega_s) \tag{4}$$

where

$$\omega_s = \frac{2\pi}{T}$$

This expression shows that the effect of combining a sampler with $F(s)$ results in a function $F^*(s)$ which is periodic in $j\omega_s$. Figure 2 shows the s-plane plots of an $F(s)$ and the corresponding $F^*(s)$.

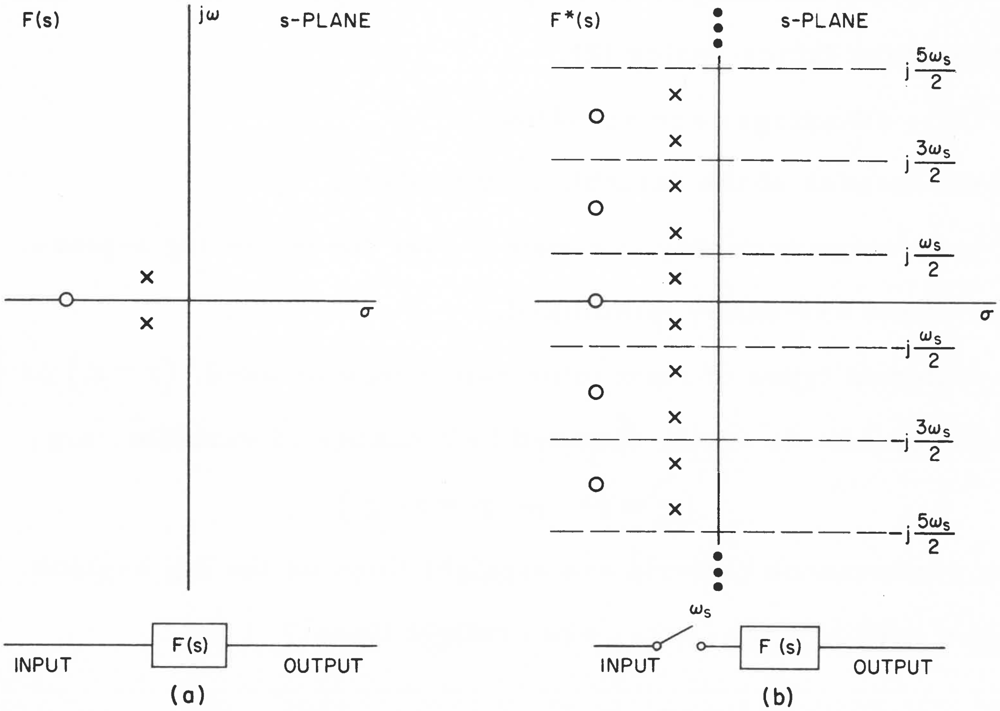


FIGURE 2 THE s-PLANE PLOTS OF A SYSTEM RESPONSE; (a) WITHOUT A SAMPLER, (b) WITH A SAMPLER

It is easily seen that $F^*(s)$ consists of an infinite number of poles and zeros which come from the repeated horizontal strips of width $j\omega_s$. In practical problems, $F^*(s)$ can be evaluated by ignoring the poles and zeros that are far removed from the σ axis. However, if $F(s)$ contains many poles and zeros, the evaluation of $F^*(s)$ will be impractical with analog methods.

The evaluation of $F^*(s)$ can be simplified by a substitution of variable (1,3,4) $z = e^{sT}$ which gives,

$$F^*(s) = F(z) \tag{5}$$

This change of variable maps one strip of width $j\omega_s$ on the s-plane into the whole z-plane. Consequently, $F(z)$ has the same number of poles as $F(s)$ and approximately the same number of zeros. $F(s)$ can now be evaluated by either a geometric or a potential analog of the z-plane and, if desired, the results can be transformed back into the s-plane or the time domain. This inverse transformation is usually not necessary because the usual criteria of stability, damping ratio, minimum settling time, and frequency response can be determined directly from the z-plane.

There are certain advantages in using the log z-plane instead of the z-plane with a potential-plane factor analog (2).

Some of these advantages are as follows:

(1) Several decades of the variable may be used.

(2) The percentage accuracy is constant over the whole log z-plane.

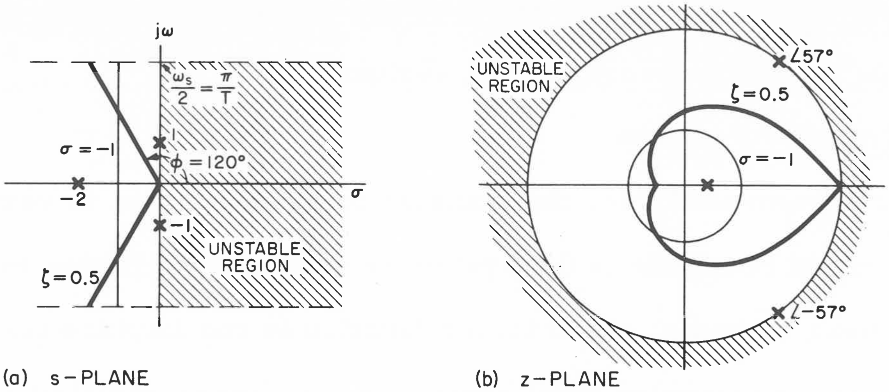
(3) Edge effects are easily minimized.

(4) Two different types of input information may be used, $(z - z_i)$ or $\left(1 - \frac{z}{z_i}\right)$.

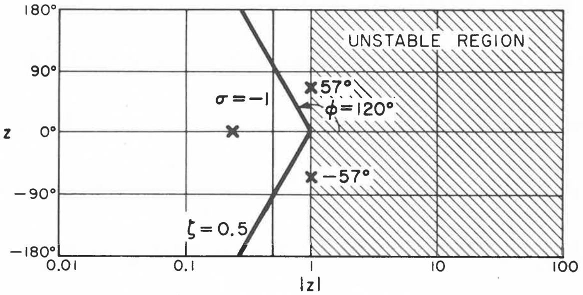
(5) Scale expansion is easily obtained by a change of variable, e.g.

$$(z' = z^2 \quad \text{or} \quad z' = z - z_i)$$

(6) Most performance criteria are straight lines on the log z-plane. (Damping ratio, minimum settling time, etc., are straight lines.)



(a) s-PLANE (b) z-PLANE



(c) LOG z-PLANE

FIGURE 3 A COMPARISON OF (a) THE s-PLANE, (b) THE z-PLANE, AND (c) THE LOG z-PLANE

Figure 3 shows the relationships between the s-plane, z-plane, and log z-plane. The left half of the s-plane maps into the area inside the unit circle in the z-plane and into the area to the left of the $|z| = 1$ line in the log z-plane. A line of constant damping ratio ζ in the s-plane maps into a heart-shaped curve in the z-plane and into a straight line in the log z-plane.

A derivation of the simple relationship between the s-plane and the log z-plane is shown below:

$$z = \epsilon^{sT} = \epsilon^{|s|T\epsilon^{j\angle s}} = \epsilon^{(\sigma + j\omega)T} \tag{6}$$

$$\mathrm{Log}_{\epsilon}(z) = sT = |s|T\epsilon^{j\angle s} = \sigma T + j\omega T \tag{7}$$

where T is the sampling interval which is a positive real number. Equation (7) shows that rectangular coordinates on the log z-plane are the same as the rectangular coordinates on the s-plane with the scales multiplied by T. Consequently, the s-plane and log z-plane plots will differ in appearance only by a scale factor T. Caution must be exercized with respect to the poles or zeros at $s = -\infty$ since these poles or zeros will appear at the origin of the log z-plane. These poles or zeros must be taken into account when evaluating $F(z)$.

The Pulse Transfer Function:

In linear system analysis, the transfer function concept is very useful for relating the output response of the system to the input excitation to the system. For continuous linear systems, the transfer function is the Laplace transform of the output response with an impulse input excitation. In linear sampled-data systems, there is an analogous concept which is called a pulse transfer function. The pulse transfer function relates the output pulse sequence to the input pulse sequence and is valid only at the sampling instants. The pulse transfer function gives no indication of the behavior of the system between sampling instants.

The Z-Transform:

In linear system problems, a continuous function of time $f(t)$ can be transformed into an algebraic function of a complex variable $F(s)$ by using the Laplace transform concept. In linear sampled-data problems, a discrete function of time $f^*(t)$ can be transformed into a function of a complex variable $F(z)$ by using an analogous concept which is the Z-transform. There is also a relation between $F(s)$ and $F(z)$ where $z = e^{sT}$, such that $F(z)$ can be found from a given $F(s)$ and vice versa. Tables of Z-transforms are available in the literature (3,4) which usually show three columns, $f(t)$, $F(s)$, and $F(z)$.

For Example:

$f(t)$	$F(s)$	$F(z)$
ϵ^{-at}	$\frac{1}{s+a}$	$\frac{1}{1-\frac{\epsilon^{aT}}{z}}$
$u(t)$	$\frac{1}{s}$	$\frac{1}{1-\frac{1}{z}}$

The procedure for finding the Z-transform $F(z)$ from a given $F(s)$ is outlined below:

(a) Expand the $F(s)$ by a partial fraction expansion to get

$$F(s) = K\left[\frac{A}{(s-s_1)} + \frac{B}{(s-s_2)} + \frac{C}{(s-s_3)} + \cdots\right] \tag{8}$$

(b) Evaluate the residues A, B, C, $\cdots$, (these can be found with a potential-plane factor analog, or often by inspection).

(c) Using the same residues, convert the s-poles to z-poles by letting $z_i = \epsilon^{s_i T}$ and set

$$F(z) = K\left[\frac{A}{\left(1-\frac{\epsilon^{s_1 T}}{z}\right)} + \frac{B}{\left(1-\frac{\epsilon^{s_2 T}}{z}\right)} + \frac{C}{\left(1-\frac{\epsilon^{s_3 T}}{z}\right)} + \cdots\right] \tag{9}$$

(d) Cross-multiply to find a common denominator which will be the product of the z-poles.

(e) Factor the numerator to get the z-zeros. (A potential-plane analog can be used for this operation).

The result will be a function $F(z)$ in the form

$$F(z) = K\,\frac{\prod_z \left(1-\frac{z_z}{z}\right)^{n_z}}{\prod_p \left(1-\frac{z_p}{z}\right)^{n_p}} \tag{10}$$

If an ESIAC® factor analog computer (2) is used, factors of the form $(z-z_i)$ or $\left(1-\frac{z}{z_i}\right)$ are more desirable than the form $\left(1-\frac{z_i}{z}\right)$ shown in Equation (10). The factor type $(z-z_i)$ can be obtained from Equation (10) by multiplying by

$$\frac{\prod_z \frac{z^{n_z}}{z^{n_z}}}{\prod_p \frac{z^{n_p}}{z^{n_p}}}$$

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$$F(z) = K \frac{\prod_z \frac{1}{z^{n_z}} (z - z_z)^{n_z}}{\prod_p \frac{1}{z^{n_p}} (z - z_p)^{n_p}} \tag{11}$$

The effect of the change in factor type results in a change in the order of the pole or zero at the origin. If all of the z-poles are first order, the result of the factor change adds a zero at the origin. This can be easily seen by factoring out a z in Equation (9).

An example of the use of the Z-transform is shown in the derivation of $G(z)$ in the Appendix.

ILLUSTRATIVE EXAMPLE:

A simple example of a sampled-data control system is shown in Figure 1. In the following discussion, this system is analyzed with an ESIAC potential-plane analog to show the effect of adding a compensating network. The system response without the compensating network is found first as shown below.

The pulse transfer function $G(z)$ is found from the corresponding $G(s)$ by using the theory of the Z-transform as shown in the Appendix.

From the Appendix:

$$G(z) = \frac{0.368K\,(z + 0.72)}{(z-1)\,(z-0.368)} \tag{12}$$

Then the closed-loop response is

$$\frac{C(z)}{R(z)} = \frac{G(z)}{1+G(z)} = \frac{0.368K\,(z+0.72)}{(z-1)\,(z-0.368) + 0.368K\,(z+0.72)} \tag{13}$$

The poles of $\frac{C(z)}{R(z)}$ are found by factoring the denominator of Equation (13).

First set the denominator equal to zero and rearrange as follows:

$$(z-1)(z-0.368) + 0.368K(z+0.72) = 0 \tag{14}$$

or

$$\frac{0.368K(z+0.72)}{(z-1)(z-0.368)} = -1 \tag{15}$$

This equation is easily solved by plotting a root-locus plot on the log z-plane with an ESIAC potential-plane analog as shown in Figure 4.

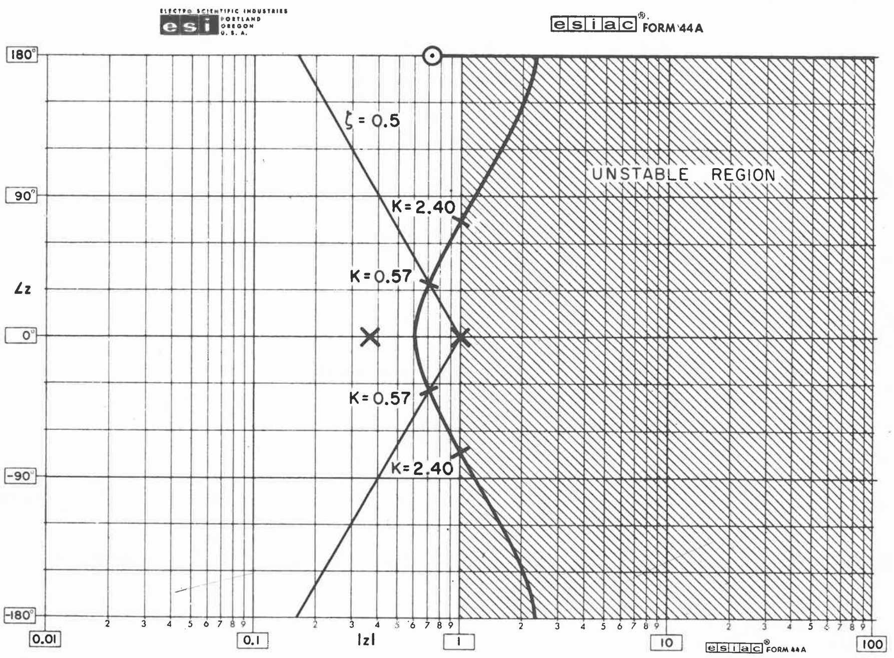

FIGURE 4 ROOT LOCUS PLOT ON THE LOG-z-PLANE. SAMPLED DATA SYSTEM WITHOUT COMPENSATION

The unstable region where $|z| > 1$ corresponds to the area outside of the unit circle on the z-plane. For a damping ratio of 0.5, $K = 0.57$ and the roots of Equation (15) are

$$z = 0.72\angle 36°,\ 0.72\angle -36°$$

The maximum value of K_m is found at the point where the loci crosses into the unstable region. At this value, $K_m = 2.40$, the system will begin to oscillate.

This system may be improved by adding a compensating network which, in effect, changes the root locus curve to some more desirable shape. For Example: it may be desirable to obtain more gain for the same value of damping ratio.

The effect of adding a pulsed lead network to the system can be shown by adding a pole-zero pair to Figure 4 and replotting the root loci. Let us add a zero at $z = 0.5\angle 0^\circ$ and a pole at $z = 0.1\angle 0^\circ$. The $F(z)$ shown in Figure 1 is then

$$F(z) = \frac{(0.2)(z-0.5)}{(z-0.1)} \tag{16}$$

A physical network that gives this $F(z)$ can be found by using the method shown in Reference 3, pages 136-43.

The closed-loop response with the pulsed lead network added is

$$\frac{C(z)}{R(z)} = \frac{F(z)\ G(z)}{1+F(z)\ G(z)} \tag{17}$$

and the characteristic equation is

$$F(z)G(z) = -1 \tag{18}$$

or

$$\frac{K(0.2)(0.368)(z-0.5)(z+0.72)}{(z-0.1)(z-0.368)(z-1)} = -1 \tag{19}$$

The root locus plot of the above equation is shown in Figure 5. For a damping ratio of 0.5, $K=5.2$ compared to $K=0.57$ for the uncompensated system. This

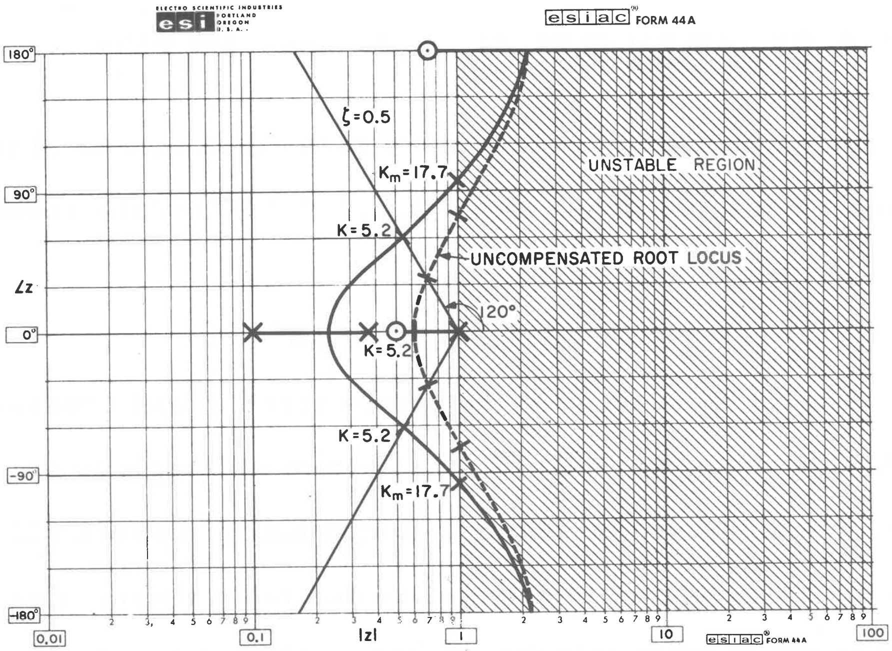

FIGURE 5 ROOT LOCUS PLOT ON THE LOG-z-PLANE. SAMPLED DATA SYSTEM WITH COMPENSATION

increase in gain corresponds to an improvement in system performance since the system error is inversely proportional to open loop gain.

The closed-loop pole locations are found from Figure 5 and are at

$$z = 0.54\angle 61°, \quad 0.54\angle -61°, \quad 0.61\angle 0°$$

The closed-loop transfer function is

$$\frac{C(z)}{R(z)} = \frac{(0.2)(0.368)(5.2)(z-0.5)(z+0.72)}{(z-0.54\angle 61°)(z-0.54\angle -61°)(z-0.61)} \tag{20}$$

Curves of the magnitude and angle of $\frac{C(z)}{R(z)}$ versus log frequency can be plotted if desired. This corresponds to the Bode plots for continuous systems in which $s = j\omega$. The data for the log frequency plots is obtained by evaluating $\frac{C(z)}{R(z)}$ for $|z| = 1$ and $\angle z$ varied from 0 to 180°. The relationship between real frequency f and $\angle z$ is

$$\angle z = \omega T = \frac{2\pi f}{f_s} \tag{21}$$

The data for the log frequency plots will be periodic with respect to one-half of the sampling frequency. This illustrates the ambiguity that results when the sampling frequency is not at least twice the highest frequency of interest in the control system.

SUMMARY:

Some fundamental concepts of sampled-data systems are presented, and the relationships between the s-plane, the z-plane, and the log z-plane are shown. The advantages of the log z-plane over the z-plane are demonstrated by an illustrative example calculated with the aid of a potential-plane factor analog computer.

APPENDIX

EXAMPLE OF Z-TRANSFORM CALCULATION

Given: $$G(s) = \frac{1-\epsilon^{-sT}}{s}\left[\frac{K}{s(s+a)}\right]$$

Required:

Find the Z-transform of $G(s)$.

First expand $G(s)$ into a partial fraction expansion and get

$$G(s) = \left(1-\epsilon^{-sT}\right)\left[\frac{A}{s^2}+\frac{B}{s}+\frac{C}{s+a}\right]$$

The coefficients A, B, C, can be evaluated with the ESIAC®, but for this simple example they can be found by inspection.

$$G(s) = \left(1-\epsilon^{-sT}\right)\left[\frac{K}{as^2}-\frac{K}{a^2 s}+\frac{K}{a^2(s+a)}\right]$$

Now take the Z-transform of each term individually and get: (see Reference 3 or 4 for Z-transform table)

$$G(z) = \left(1-\frac{1}{z}\right)\left[\frac{K}{a}\,\frac{Tz}{(z-1)^2}-\frac{K}{a^2}\,\frac{z}{(z-1)}+\frac{K}{a^2}\,\frac{z}{z-\epsilon^{-aT}}\right]$$

where $z = \epsilon^{sT}$ and $1-\frac{1}{z} = \frac{z-1}{z}$

Then $$G(z) = \frac{K}{a}\left[\frac{T}{(z-1)}-\frac{1}{a}+\frac{z-1}{a\left(z-\epsilon^{-aT}\right)}\right]$$

Now find a common denominator for the term in brackets and get:

$$G(z) = \frac{K}{a}\left[\frac{Ta\left(z-\epsilon^{-aT}\right)+(z-1)\left(\epsilon^{-aT}-1\right)}{a(z-1)\left(z-\epsilon^{-aT}\right)}\right]$$

For this problem, $T = a = 1$, and $G(z)$ reduces to

$$G(z) = \frac{\frac{K}{\epsilon}(z+0.72)}{(z-1)\left(z-\epsilon^{-aT}\right)} = \frac{0.368K(z+0.72)}{(z-1)\left(z-\epsilon^{-aT}\right)}$$

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